# Invariants of Conformal and Projective Structures

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Dedicated to Katsumi Nomizu

**Abstract**

While in Euclidean, equiaffine or centroaffine differential geometry there exists a *unique*, distinguished normalization of a regular hypersurface immersion $x : M^n \to A^{n+1}$, in the geometry of the general affine transformation group, there only exists a distinguished *class* of such normalizations, the class of *relative normalizations*. Thus, the appropriate invariants for speaking about affine hypersurfaces are *invariants of the induced classes*, e.g. the conformal class of induced metrics and the projective class of induced conormal connections. The aim of this paper is to study such invariants. As an application, we reformulate the fundamental theorem of affine differential geometry.

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All connections that appear in this paper shall be assumed as *torsionless* and *admitting a parallel volume form.* As a general reference, see [S-S-V 91].

## 1 Conjugacy

**Definition 1.1** *Let $(M, h)$ be a semi-riemannian manifold, $\nabla, \nabla^\star$ connections on $M$. Then the triple $\{\nabla, h, \nabla^\star\}$ is called* conjugate *if*

$$\forall u, v, w : uh(v, w) = h(\nabla_u v, w) + h(v, \nabla^\star_u w).$$

Note that $h$ and one of the connections uniquely determine the other, 'conjugate' connection.

For the main theorems, we will need the following three results about conjugate triples: (The proofs can be found in [No-Si 91].)

**Lemma 1.2** *(Characterization of Conjugacy)*
*Let $h, \nabla, \nabla^\star$ as above. Then*

$$\{\nabla, h, \nabla^\star\} \text{ conjugate } \Leftrightarrow \begin{cases} \frac{1}{2}(\nabla + \nabla^\star) = \nabla(h) \\ \{\nabla, h\} \text{ is a Codazzi pair}^1, \end{cases}$$

*where $\nabla(h)$ shall denote the Levi-Cività-connection corresponding to $h$.*

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[1] $\nabla$ and $h$ satisfy Codazzi equations.

**Lemma 1.3** *(Conjugacy and Curvature)*
*Let $\{\nabla, h, \nabla^\star\}$ be a conjugate triple. Then the following conditions hold:*

*a)* $\forall u, v, w, z : h(R(u,v)w, z) + h(w, R^\star(u,v)z) = 0,$

*b)* $trace_h Ric = trace_h Ric^\star =: r,$

*where $R, R^\star, Ric, Ric^\star$ are the Riemannian curvature tensors and Ricci tensors corresponding to $\nabla, \nabla^\star$, resp.*

**Lemma 1.4** *(Conjugate Codazzi Equations)*
*Let $\{\nabla, h, \nabla^\star\}$ be a conjugate triple, $\widehat{S}$ a bilinear form on $M$. Then*

$$(\{\nabla^\star, \widehat{S}\} \text{ is a Codazzi pair}) \Leftrightarrow (\{\nabla, S\} \text{ is a Codazzi pair}),$$

*where $S$ is implicitly defined by $\forall v, w : h(Sv, w) = \widehat{S}(v, w)$. (We also write $S = \widehat{S}^h$.)*

# 2 Conformal and Projective Structures

**Definition 2.1** *Let $M$ be a differentiable manifold.*

*a)* *Let $f, f^\sharp \in C^\infty(M)$. Define $f \sim f^\sharp :\Leftrightarrow f - f^\sharp = const.$*
$\tau \in C^\infty(M)/_\sim$ *is called* transformation function.

*b)* *Let $h, h^\sharp$ be semi-riemannian metrics on $M$. They are called* conformally equivalent *if* $h^\sharp = exp(2\tau)h$ *for some transformation function $\tau$.*
*Notation:* $\mathcal{C}(h) := \{h^\sharp \mid h, h^\sharp$ *conformally equivalent* $\}$.

*c)* *Let $\nabla^\star, \nabla^{\star\sharp}$ be connections on $M$. They are called* strongly projectively equivalent *if* $\forall v, w : \nabla^{\star\sharp}_v w = \nabla^\star_v w + 2d\tau(v)w + 2d\tau(w)v$ *for some transformation function $\tau$.*
*Notation:* $\mathcal{P}(\nabla^\star) := \{\nabla^{\star\sharp} \mid \nabla^\star, \nabla^{\star\sharp}$ *strongly projectively equivalent* $\}$.

*d)* *Let $\mathcal{C}$ as above, $\nabla, \nabla^\sharp$ be connections on $M$. They are called* conprojectively equivalent *if* $\forall v, w : \nabla^\sharp_v w = \nabla_v w - 2h(v,w) grad_h \tau$ *for some transformation function $\tau$.*
*Notation:* $\mathcal{K}_{\mathcal{C}}(\nabla) := \{\nabla^\sharp \mid \nabla, \nabla^\sharp$ *conprojectively equivalent* $\}$.

**Remark 2.2** Usually, two connections $\nabla^\star, \nabla^{\star\sharp}$ are called *projectively equivalent* if there exists a one-form $\theta$ such that $\forall v, w : \nabla^{\star\sharp}_v w = \nabla^\star_v w + \theta(v)w + \theta(w)v$. However, if both connections have *symmetric Ricci tensors* (or, equivalently, they admit parallel volume forms, which was our general assumption), then the one-form $\theta$ satisfies $d\theta = 0$, which means that, locally, they are strongly projectively equivalent in the sense of the definition above. If, additionally, the manifold $M$ is *simply connected*, we have $\theta = 2d\tau$ globally for some transformation function $\tau$.

**Motivation** *for the choice of the notion 'conprojective'*
Let $\{\nabla, h, \nabla^\star\}$ and $\{\nabla^\sharp, h^\sharp, \nabla^{\star\sharp}\}$ be conjugate triples. The metrics shall satisfy $h^\sharp = exp(2\tau)h$ for some transformation function $\tau$. Then:

$$\left(\nabla^\sharp_v w = \nabla_v w - 2h(v,w) grad_h \tau\right) \Leftrightarrow \left(\nabla^{\star\sharp}_v w = \nabla^\star_v w + 2d\tau(v)w + 2d\tau(w)v\right),$$

so $\nabla^\sharp, \nabla$ are conprojectively equivalent iff their conjugate connections $\nabla^{\star\sharp}, \nabla^\star$ are strongly projectively equivalent.
For further results about conprojective equivalence, see [Iva-94].

**Lemma 2.3** *(Conjugacy as an Invariant)*
*Let $\{\nabla, h, \nabla^\star\}$ be a (not necessarily conjugate) triple and let $\tau$ be a transformation function. $\{\nabla^\sharp, h^\sharp, \nabla^{\star\sharp}\}$ shall denote the triple we get if we change $\{\nabla, h, \nabla^\star\}$ with $\tau$ according to 2.1. Then*

$$(\{\nabla, h, \nabla^\star\} \text{ is a conjugate triple}) \Leftrightarrow (\{\nabla^\sharp, h^\sharp, \nabla^{\star\sharp}\} \text{ is a conjugate triple}).$$

*In this case of conjugacy, let $Con(\mathcal{K}_\mathcal{C}, \mathcal{C}, \mathcal{P})$ denote the class of conjugate triples we get from $\mathcal{C}(h), \mathcal{P}(\nabla^\star)$ and $\mathcal{K}_\mathcal{C}(\nabla)$.*

The proof is a simple calculation.

# 3 Affine Hypersurfaces

In this section, we recall some basic facts from affine hypersurface theory. Let $M$ here be a connected, simply connected differentiable manifold. Let $x : M^n \to A^{n+1}$ be a regular[2] hypersurface immersion and $\{Y, y\}$ a relative normalization of $x$.

**Definition 3.1** *(Structure Equations)*
*The geometry of $A^{n+1}$ induces, via the triple $\{x, Y, y\}$, the following quantities on $M$:*

$$\begin{aligned} \overline{\nabla}_v dx(w) &= dx(\nabla_v w) + h(v,w)y \\ dy(w) &= -dx(Sw) \\ \overline{\nabla}_v dY(w) &= dY(\nabla^\star) - \widehat{S}(v,w)Y. \end{aligned}$$

**Lemma 3.2** *The induced quantities have the following properties:*

*a) $h$ is a semi-riemannian metric,*

*b) $\nabla, \nabla^\star$ are torsionless connections which admit parallel volume forms,*

*c) $\{\nabla, h, \nabla^\star\}$ is a conjugate triple,*

*d) $(n-1)\widehat{S} = Ric^\star$ and $h(Sv,w) = h(v,Sw) = \widehat{S}(v,w)$.*

**Remark 3.3** If $\{Y, y\}$ is a relative normalization of the hypersurface $x$, let $\{Y^\sharp, y^\sharp\}$ be another one with $Y^\sharp = exp(2\tau)Y$. Then the corresponding induced conjugate triples $\{\nabla, h, \nabla^\star\}, \{\nabla^\sharp, h^\sharp, \nabla^{\star\sharp}\}$ change as in 2.1. So a regular affine hypersurface induces a structure of the type $Con(\mathcal{K}_\mathcal{C}, \mathcal{C}, \mathcal{P})$.

[2] A hypersurface immersion $x$ into affine space is called *regular* if, for an arbitrary conormal field $Y$, the bilinear form $\langle dx, dY\rangle$ is nondegenerate.

In 1990, Dillen, Nomizu and Vrancken formulated the following version of the fundamental theorem of (equi-)affine geometry (see [D-N-V 90]):

**Theorem 3.4** *(Existence of Affine Hypersurfaces)*
*Let $M$ be a manifold as above, $h$ a semi-riemannian metric on $M$ and $\nabla$ a connection on $M$. Assume that $\{\nabla, h\}$ is a Codazzi pair. $\nabla^\star$ shall denote the connection that makes $\{\nabla, h, \nabla^\star\}$ a conjugate triple. Then the following conditions are equivalent:*

*a) There exists a hypersurface immersion $x : M^n \to A^{n+1}$ together with a relative normalization $\{Y, y\}$ which induce $\{\nabla, h, \nabla^\star\}$ as in 3.1.*

*b) $\nabla^\star$ is projectively flat.*

# 4 Invariants of $Con(\mathcal{K}_\mathcal{C}, \mathcal{C}, \mathcal{P})$

We now study some invariants of the structures we introduced in section 2. $\{\nabla, h, \nabla^\star\}$ shall denote an arbitrary representative of the class $Con(\mathcal{K}_\mathcal{C}, \mathcal{C}, \mathcal{P})$.

**Definition 4.1** *(Flatness of $Con(\mathcal{K}_\mathcal{C}, \mathcal{C}, \mathcal{P})$)*
*From 1.3 we know that, for a conjugate triple $\{\nabla, h, \nabla^\star\}$, $\nabla$ is a flat connection iff $\nabla^\star$ is one. If in $Con(\mathcal{K}_\mathcal{C}, \mathcal{C}, \mathcal{P})$ there exists such a triple with flat connections, we say that $Con(\mathcal{K}_\mathcal{C}, \mathcal{C}, \mathcal{P})$ is* flat.

Of course, flatness is by definition an invariant of $Con(\mathcal{K}_\mathcal{C}, \mathcal{C}, \mathcal{P})$.

**Lemma 4.2** *(Conjugation of* Weyl*'s Projective Curvature Tensor)*
*The following tensor $K_1$ is an invariant of $Con(\mathcal{K}_\mathcal{C}, \mathcal{C}, \mathcal{P})$:*

$$\forall u, v, w : K_1(u, v)w := R(u, v)w - \frac{1}{n-1}(h(v, w)Ric^{\star h}(u) - h(u, w)Ric^{\star h}(v)).$$

**Proof:** Just check with 1.3 that

$$h(K_1(u, v)w, z) + h(w, P^\star(u, v)z) = 0,$$

where the *projective curvature tensor* $P^\star$ is given by

$$\forall u, v, w : P^\star(u, v)w := R^\star(u, v)w - \frac{1}{n-1}(Ric^\star(v, w)u - Ric^\star(u, w)v).$$

It is well-known that $P^\star$ is an invariant of $\mathcal{P}$.

**Lemma 4.3** *(See Kurose in [Kur 91])*
*The tensor $K_2$ defined by:*

$$\begin{aligned}\forall u, v, w :\ K_2(u, v)w := R(u, v)w &+ (h(v, w)Ric^h(u) - h(u, w)Ric^h(v)) \\ &- \frac{r}{n-1}(h(v, w)u - h(u, w)v)\end{aligned}$$

*is an invariant of $Con(\mathcal{K}_\mathcal{C}, \mathcal{C}, \mathcal{P})$.*

For the proof, calculate the transformation of the curvature tensors.

The tensor $K_2$ is the traceless part of the tensor $K_1$. More precisely, we have the following results:

**Lemma 4.4** *(The Invariant Bilinear Form)*
*Let $K_1, K_2$ as above. Define*

$$\forall u, v : B(u, v) := (n-1)Ric(K_1)(u, v) := (n-1)trace(z \mapsto K_1(z, u)v).$$

*Then the following holds:*

*a)* $B = (n-1)Ric + Ric^{\star} - rh$,

*b)* $B$ *is an invariant of* $Con(\mathcal{K}_{\mathcal{C}}, \mathcal{C}, \mathcal{P})$,

*c)* $(B = 0) \Leftrightarrow (K_1 = K_2)$,

*d)* $(Con(\mathcal{K}_{\mathcal{C}}, \mathcal{C}, \mathcal{P})$ *flat*$) \Rightarrow (B \equiv 0)$,

*e) in general,* $B \not\equiv 0$.

**Proof:** a),b),c), and d): straightforward. For proving e), consider the conjugate triple $\{\nabla(h), h, \nabla(h)\}$, where $h$ is an arbitrary semi-riemannian metric on $M$. From a) we immediately see that $B \equiv 0$ means that $(M, h)$ is an Einstein space, which generally is not true.
**Problem:** It is still an open question whether the converse of part d) in the preceding lemma holds: does the identical vanishing of the bilinear form $B$ imply the flatness of $Con(\mathcal{K}_{\mathcal{C}}, \mathcal{C}, \mathcal{P})$? Of course, this could only be true for dimensions $n \geq 3$, for in dimension two, we always have $B \equiv 0$.

# 5 The Fundamental Theorem of Affine Hypersurface Theory

We now want to reformulate the fundamental theorem 3.4 for affine hypersurfaces without referring to any particular normalization. A first version of this type was given in [Si 92].

**Theorem 5.1** *(Fundamental Theorem, reformulated)*
*Let $M$ be a connected, simply connected differentiable manifold. On $M$, a class $Con(\mathcal{K}_{\mathcal{C}}, \mathcal{C}, \mathcal{P})$ of conjugate triples shall be given. Then the following conditions are equivalent:*

*a) There exists a hypersurface immersion $x : M^n \to A^{n+1}$ which induces $Con(\mathcal{K}_{\mathcal{C}}, \mathcal{C}, \mathcal{P})$ as described in 3.3.*

*b) $Con(\mathcal{K}_{\mathcal{C}}, \mathcal{C}, \mathcal{P})$ is flat.*

**Proof:** Combine theorem 3.4 with remark 3.3 and the definition 4.1 of flatness.

The problem now is to find characterizations for flatness of $Con(\mathcal{K}_{\mathcal{C}}, \mathcal{C}, \mathcal{P})$.

**Theorem 5.2** *(First Characterization)*
*Let $Con(\mathcal{K}_{\mathcal{C}}, \mathcal{C}, \mathcal{P})$ be given on $M$. Then*

$$Con(\mathcal{K}_{\mathcal{C}}, \mathcal{C}, \mathcal{P}) \text{ is flat } \Leftrightarrow \begin{cases} a) & K_1 \equiv 0, \\ b) & \{\nabla, Ric^{\star h}\} \text{ is a Codazzi pair.} \end{cases}$$

**Proof:** Conjugate *Weyl's* well-known characterization of projective flatness, use lemma 1.4 and the definition of $K_1$.

**Theorem 5.3** *(Second Characterization,* Kurose*)*
*Let* $Con(\mathcal{K}_\mathcal{C}, \mathcal{C}, \mathcal{P})$ *be given on* $M$*. Then*

$$Con(\mathcal{K}_\mathcal{C}, \mathcal{C}, \mathcal{P}) \text{ is flat } \Leftrightarrow \left\{ \begin{array}{ll} a) & K_2 \equiv 0, \\ b) & \{\nabla, \frac{r}{n-1} Id - Ric^h\} \text{ is a Codazzi pair.} \end{array} \right.$$

**Proof:** Calculate the integrability conditions for the existence of a flat connection $\nabla^\sharp$ that is conprojectively equivalent to $\nabla$.

**Remark 5.4** In the characterization theorems above, we have the following relations between the conditions a) and b):
$n = 2$ : condition a) holds trivially,
$n > 2$ : condition a) implies condition b).

# References


[S-S-V 91] Udo Simon, Angela Schwenk-Schellschmidt and Helmut Viesel: *Introduction to the Affine Differential Geometry of Hypersurfaces*, Lecture Notes, Science University of Tokyo (1991); ISBN 3798315299

[Kur 91] Takashi Kurose: *On the Realization of Statistical Manifolds in Affine Space*, preprint Fukuoka University (1991)

[No-Si 91] Katsumi Nomizu and Udo Simon: *Notes on Conjugate Connections*, in: Geometry and Topology of Submanifolds, IV (eds.: Franki Dillen and Leopold Verstraelen), 152-173, World Scientific Singapore etc. (1991)

[Iva-94] Stefan Ivanov: *On Dual-Projectively Flat Affine Connections*, preprint University of Sofia (1994)

[Si 92] Udo Simon: *Conformal and Projective Structures on Ovaloids*, in: Geometry and Topology of Submanifolds, V (eds.: Franki Dillen, Leopold Verstraelen, Luc Vrancken en Ignace van de Woestijne), 254-259, World Scientific Singapore etc. (1992)

[D-N-V 90] Franki Dillen, Katsumi Nomizu and Luc Vrancken: *Conjugate Connections and Radon's Theorem in Affine Differential Geometry*, Monatsheft fr Mathematik 109 (1990), 221-235



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